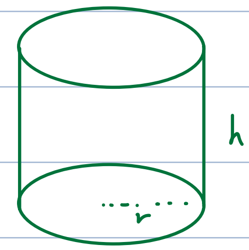
# MATH2010E TUTO 12

**9. Minimum surface area with fixed volume** Find the dimensions of the closed right circular cylindrical can of smallest surface area whose volume is $16\pi$ cm$^3$.

Ans: Minimize $S(r,h) := 2\pi r h + 2\pi r^2$
under constraint $V(r,h) := \pi r^2 h = 16\pi$ } both $C^1$




Then $\nabla S = (2\pi h + 4\pi r, 2\pi r)$
$\nabla V = (2\pi r h, \pi r^2) \neq \vec{0}$

Consider $F(r,h,\lambda) = 2\pi r h + 2\pi r^2 - \lambda(\pi r^2 h - 16\pi)$

$$\begin{cases} 0 = \frac{\partial F}{\partial r} = 2\pi h + 4\pi r - 2\pi \lambda r h & ① \\ 0 = \frac{\partial F}{\partial h} = 2\pi r - \lambda \pi r^2 & ② \\ 0 = -\frac{\partial F}{\partial \lambda} = \pi r^2 h - 16\pi & ③ \end{cases}$$

② : $\pi(2r - \lambda) = 0$

$r = 0$ or $r = \frac{2}{\lambda}$
(impossible)

① : $h + 2r = \frac{2}{r}(rh) = 2h$

$\Rightarrow h = 2r$

③ : $r^3 = 8$

$\Rightarrow r = 2, \quad h = 4$

So $(r,h) = (2,4)$ is the only critical pt.
and $S(2,4) = 24\pi$

Hence radius $= 2$ cm, height $= 4$ cm give the smallest surface area $24\pi$ cm$^2$ //

**20. Minimum distance to the origin** Find the point on the surface $z = xy + 1$ nearest the origin.

Ans Minimize $f(x,y,z) := x^2 + y^2 + z^2$

under constraint $g(x,y,z) := z - xy - 1 = 0$

$$\nabla f = (2x, 2y, 2z)$$

$$\nabla g = (-y, -x, 1) \neq \vec{0}$$

Consider $F(x,y,z,\lambda) = x^2 + y^2 + z^2 - \lambda(z - xy - 1)$

$$\begin{cases} 0 = \frac{\partial F}{\partial x} = 2x + \lambda y & ① \\ 0 = \frac{\partial F}{\partial y} = 2y + \lambda x & ② \\ 0 = \frac{\partial F}{\partial z} = 2z - \lambda & ③ \\ 0 = -\frac{\partial F}{\partial \lambda} = z - xy - 1 & ④ \end{cases}$$

①, ②: $4x = (-\lambda)^2 x$

$(4 - \lambda^2) x = 0$

Case 1 : $x = 0$. $\Rightarrow y = 0 \Rightarrow z = 1$

Case 2 : $\lambda = 2 \Rightarrow x = -y, \; z = 1 \Rightarrow y^2 = 0$

$\Rightarrow (x, y, z) = (0, 0, 1)$

Case 3 : $\lambda = -2 \Rightarrow x = y, \; z = -1$

$\Rightarrow x^2 = -2$ no sol.

Since $(0,0,1)$ is the only critical pt

and there is no max. distance from the surface to the origin,

so $(0,0,1)$ is the pt. on the surface closest to the origin //

**33. Maximizing a utility function: an example from economics** In economics, the usefulness or *utility* of amounts $x$ and $y$ of two capital goods $G_1$ and $G_2$ is sometimes measured by a function $U(x, y)$. For example, $G_1$ and $G_2$ might be two chemicals a pharmaceutical company needs to have on hand and $U(x, y)$ the gain from manufacturing a product whose synthesis requires different amounts of the chemicals depending on the process used. If $G_1$ costs $a$ dollars per kilogram, $G_2$ costs $b$ dollars per kilogram, and the total amount allocated for the purchase of $G_1$ and $G_2$ together is $c$ dollars, then the company's managers want to maximize $U(x, y)$ given that $ax + by = c$. Thus, they need to solve a typical Lagrange multiplier problem.

Suppose that

$$U(x, y) = xy + 2x$$

and that the equation $ax + by = c$ simplifies to

$$g(x,y) := 2x + y = 30.$$

Find the maximum value of $U$ and the corresponding values of $x$ and $y$ subject to this latter constraint.

Ans:

$$\nabla U = (y+2, x)$$

$$\nabla g = (2, 1) \neq \vec{0}$$

Consider $F(x,y,\lambda) = xy + 2z - \lambda(2x+y-30)$

$$\begin{cases} 0 = \frac{\partial F}{\partial x} = y+2-2\lambda \\ 0 = \frac{\partial F}{\partial y} = x - \lambda \\ 0 = -\frac{\partial F}{\partial z} = 2x+y-30 \end{cases}$$

$$2\lambda + (2\lambda - 2) = 30$$

$$\lambda = 8$$

$$\Rightarrow (x, y) = (8, 14)$$

So $U(8,14) = 128$ is the max. value of $U$ under the constraint.

**42. a. Maximum on line of intersection** Find the maximum value of $w = xyz$ on the line of intersection of the two planes $x + y + z = 40$ and $x + y - z = 0$.

**b.** Give a geometric argument to support your claim that you have found a maximum, and not a minimum, value of $w$.

Ans: a) Let
$$\left.\begin{aligned} f(x,y,z) &= xyz \\ g_1(x,y,z) &= x+y+z \\ g_2(x,y,z) &= x+y-z \end{aligned}\right\} \text{ all } C^1$$

Maximize $f$ subject to constraints $\begin{cases} g_1 = 40 \\ g_2 = 0 \end{cases}$

$$\nabla f = (yz, xz, xy)$$
$$\left.\begin{aligned} \nabla g_1 &= (1, 1, 1) \\ \nabla g_2 &= (1, 1, -1) \end{aligned}\right\} \text{ linearly independent}$$

Consider $F(x,y,z,\lambda_1,\lambda_2) = xyz - \lambda_1(x+y+z-40) - \lambda_2(x+y-z)$

$$\begin{cases} 0 = \frac{\partial F}{\partial x} = yz - \lambda_1 - \lambda_2 & (1) \\ 0 = \frac{\partial F}{\partial y} = xz - \lambda_1 - \lambda_2 & (2) \\ 0 = \frac{\partial F}{\partial z} = xy - \lambda_1 + \lambda_2 & (3) \\ 0 = -\frac{\partial F}{\partial \lambda_1} = x+y+z-40 & (4) \\ 0 = -\frac{\partial F}{\partial \lambda_2} = x+y-z & (5) \end{cases}$$

(1), (2): $yz = xz \Rightarrow (x-y)z = 0$

Case 1: $x = y \overset{(4)(5)}{\Rightarrow} \begin{cases} 2x + z = 40 \\ 2x - z = 0 \end{cases}$

$\Rightarrow x = y = 10, \quad z = 20$

Case 2: $z = 0 \overset{(4),(5)}{\Rightarrow} \begin{cases} x+y = 40 \\ x+y = 0 \end{cases}$ No sol.

So max occurs at $(10, 10, 20)$ with value $(10)(10)(20) = 2000$ //

b) $\vec{n} := \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -2\hat{i} + 2\hat{j}$ is $\parallel$ to the line of intersection

$\Rightarrow$ The line is $\vec{x} = t(-2,2,0) + (10,10,20)$
$= (-2t+10,\ 2t+10,\ 20)$

So $w = (-2t+10)(2t+10)(20)$
$= 20(100-4t^2)$
which has no min
but has max at $t = 0 \Rightarrow (x,y,z) = (10,10,20)$ //

**48. Sum of products** Let $a_1, a_2, \ldots, a_n$ be $n$ positive numbers. Find the maximum of $\sum_{i=1}^{n} a_i x_i$ subject to the constraint $\sum_{i=1}^{n} x_i^2 = 1$.

Ans: Let $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i$

$g(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2 - 1$ } both $C^1$

Minimiz $f$ subject to $g = 0$

Consider $F(x_1, \ldots, x_n, \lambda) = \sum_{i=1}^{n} a_i x_i - \lambda\left(\sum_{i=1}^{n} x_i^2 - 1\right)$

$$\begin{cases} \text{For } 1 \le i \le n, & 0 = \frac{\partial F}{\partial x_i} = a_i - 2\lambda x_i \quad ① \\ & 0 = -\frac{\partial F}{\partial \lambda} = \sum_{i=1}^{n} x_i^2 - 1 \quad ② \end{cases}$$

①: $2\lambda x_i = a_i > 0 \Rightarrow \lambda, x_i \neq 0$

$\Rightarrow x_i = \frac{a_i}{2\lambda}$

②: $\sum_{i=1}^{n} \left(\frac{a_i}{2\lambda}\right)^2 = 1$

$\Rightarrow 4\lambda^2 = \sum_{i=1}^{n} a_i^2$

$\Rightarrow 2\lambda = \pm \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}$

Case 1: $2\lambda = \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}$

$\Rightarrow f = \sum_{i=1}^{n} a_i \cdot \left(\frac{a_i}{2\lambda}\right) = \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \leftarrow$ max.

Case 2: $2\lambda = -\left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}$

$\Rightarrow f = -\left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}$

So the max of $f$ subject to constraint $g = 0$ is $\left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}$ //